

# A SPECIALIZATION INEQUALITY FOR TROPICAL COMPLEXES

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**ABSTRACT.** We prove a specialization inequality relating the dimension of the complete linear series on a variety to the tropical complex of a regular semistable degeneration. Our result extends Baker's specialization inequality to arbitrary dimension.

## 1. INTRODUCTION

The specialization inequality for curves [Bak08] gives a bound for the dimension of a linear system on a curve in terms of an analogous combinatorial invariant on the dual graph of a degeneration. For higher dimensional varieties, understanding linear equivalence on the dual complex of a semistable degeneration requires additional information beyond the dual complex, which can be encoded in a tropical complex, as introduced in [Car13]. In this paper, we generalize the specialization inequality to varieties of arbitrary dimension using tropical complexes.

Similar to the case of curves, our specialization inequality applies to a regular, strictly semistable degeneration  $\mathfrak{X}$  over a discrete valuation ring. Thus, the special fiber of  $\mathfrak{X}$  is a reduced union of smooth varieties, with simple normal crossing intersections. From  $\mathfrak{X}$ , we can construct the dual complex  $\Delta$ , which is a regular  $\Delta$ -complex recording how the components of the special fiber intersect. In addition, the tropical complex  $\Delta$  records certain intersection numbers from  $\mathfrak{X}$ , the details of which will be recalled in Section 2. In addition, [Car13, Sec. 4] introduced both a specialization map  $\rho$  from divisors on the general fiber of  $\mathfrak{X}$  to the tropical complex  $\Delta$  as well as a compatible notion of linear equivalence on  $\Delta$ . We define  $h^0$  of a divisor on  $\Delta$  as the fewest number of points such that no linearly effective divisor contains all of these points (see Def. 3.1 for details), and then prove a specialization inequality:

**Theorem 1.1.** *Let  $\mathfrak{X}$  be a regular strictly semistable degeneration of relative dimension  $n$  over a discrete valuation ring. Further suppose that the locally closed strata of dimension at most  $n - 2$  in  $\mathfrak{X}$  are affine and that  $\mathfrak{X}$  is robust in dimensions  $n - 1$  and  $n$ . If  $D$  is any divisor on the general fiber  $X$  of  $\mathfrak{X}$ , and  $\Delta$  is the tropical complex of  $\mathfrak{X}$ , then we have:*

$$\dim H^0(X, \mathcal{O}(D)) \leq h^0(\Delta, \rho(D))$$

The second sentence of Theorem 1.1 places additional requirements on the degeneration beyond the strict semistability, whose definitions we now explain. A *closed stratum* of dimension  $n - k$  in  $\mathfrak{X}$  is a connected component of the intersection of any  $k + 1$  components of the special fiber. The *locally closed stratum* is formed by removing all lower-dimensional closed strata from a fixed closed stratum, and so Theorem 1.1 requires that these differences are affine varieties, when they have dimension at most  $n - 2$ . For the closed strata of dimension  $n - 1$  and  $n$ , Theorem 1.1 puts a weaker condition of *robustness*, which means that the union of the lower dimensional strata form a big divisor, in the sense of birational geometry [Laz04, Sec. 2.2].

Although Theorem 1.1 does not apply directly to all regular semistable degenerations because of these hypotheses, we note that any regular semistable degeneration whose special fiber has projective components can be modified into one with affine locally closed strata by a sequence of blow-ups [Car13, Prop. 2.10].

We outline the structure of the proof, which will also serve to explain the necessity of the robustness and affineness hypotheses in Theorem 1.1. Following the proof of the specialization inequality for curves, our specialization inequality essentially follows from our definition of  $h^0$  given a specialization function which preserves linear equivalence, effectivity of divisors, and point containment. Compatibility of linear equivalence was proved in [Car13]. However, the specialization  $\rho(D)$  of an effective divisor  $D$  is not always effective, so we introduce a refined specialization in Section 5, which is always effective, and is an element of the same complete linear series as  $\rho(D)$ . Note that the refined specialization is not necessarily supported on the  $(n - 1)$ -dimensional simplices of  $\Delta$ , which is one reason why it is important for us to consider divisors which are not just linear combinations of the ridges.

Although our proof does not use this technology explicitly, the refined specialization of a divisor  $D$  can be understood in terms of the projection of  $D$  to the skeleton of the Berkovich analytification. More specifically, by results going back to Berkovich [Ber90], the dual complex  $\Delta$  embeds into, and is a strong deformation retract of, the analytification of the general fiber of  $\mathfrak{X}$ . Restricting to effective divisors in order to avoid cancellation, we have:

**Proposition 1.2.** *If  $D$  is an effective divisor in the general fiber of  $\mathfrak{X}$ , then the projection of the analytification of  $D$  to the skeleton defined by  $\mathfrak{X}$  equals the support of the refined specialization of  $D$ , except on a set of codimension 2 or greater.*

While the projection of the analytification always preserves point containment, Proposition 1.2 shows that the same can fail for the refined tropicalization whenever the projection has components of codimension 2 or greater. The purpose of the robustness and affineness hypotheses in Theorem 1.1 are so that we can lift points from the tropical complex to the algebraic variety in such a way that containing divisors project to a set of codimension 1 on  $\Delta$ . Also, note that, in the case of curves, the set

of codimension 2 in Proposition 1.2 is necessarily empty, which is why the specialization inequality for curves required no hypotheses beyond a strictly semistable degeneration.

As an application of Theorem 1.1, we give an example of a tropical complex which does not lift to any algebraic variety.

**Theorem 1.3.** *There exists a 2-dimensional tropical complex  $\Delta$  which is not the tropical complex of any regular semistable degeneration.*

The underlying  $\Delta$ -complex of the example in Theorem 1.3 is a triangulation of the product of a cycle and an interval. Therefore, the dual complex is realizable from a degeneration of an algebraic surface, such as the product of an elliptic curve with a projective line. However, in Theorem 1.3 the structure constants of the tropical complex  $\Delta$  are chosen so that for any divisor  $D$ ,  $h^0(\Delta, nD)$  grows at most linearly in  $n$ , which combined with Theorem 1.1, shows that  $\Delta$  can't be the tropical complex of a projective surface.

An alternative approach to a specialization inequality has been developed in unpublished work of Eric Katz and June Huh. They work not with degenerations, but with tropicalizations of surfaces embedded in  $\mathbb{G}_m^N$  over a field with trivial valuation. Moreover, their proof involves choosing linearly equivalent divisors passing through the lowest dimensional toric strata, as opposed to the proof of Theorem 1.1 in which points are lifted from the  $n$ -dimensional locally closed strata. Therefore, the two approaches should give distinct and possibly complementary bounds on the dimension of linear series.

The rest of this paper is organized as follows. Section 2 recalls the definition of tropical complexes from [Car13] as well as their main properties. In Section 3, we define the invariant  $h^0$  from Theorem 1.1 and look at some examples and applications. Section 4 proves compatibility of tropical complexes under ramified base changes followed by toroidal resolutions of singularities, which are the tools used to give a refined specialization in Section 5. Section 6 has the proof of the specialization inequality.

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## 2. TROPICAL COMPLEXES

A tropical complex is built from a  $\Delta$ -complex, which is a combinatorially defined class of topological spaces [Hat02, Sec. 2.1]. This paper will only deal with tropical complexes coming from strictly semistable degenerations,

which lead to regular  $\Delta$ -complexes, meaning that the faces of a fixed simplex are always distinct. We recall the definition of a tropical complex in the regular case.

**Definition 2.1** (Def. 2.1 in [Car13]). Let  $n$  be an integer and let  $\Delta$  be a connected, finite, regular  $\Delta$ -complex, whose simplices all have dimension at most  $n$ . We will refer to the  $n$ - and  $(n - 1)$ -dimensional simplices of  $\Delta$  as *facets* and *ridges* respectively. A *weak tropical complex*, also denoted  $\Delta$ , consists of  $\Delta$  together with structure constants, which are integers  $\alpha(v, r)$  for every vertex  $v$  of every ridge  $r$  of  $\Delta$ , such that, for each ridge  $r$ , we have the equality:

$$(1) \quad \sum_{v \in r} \alpha(v, r) = \deg r,$$

where  $\deg r$  is defined to be the number of facets containing  $r$ .

If  $q$  is an  $(n - 2)$ -dimensional simplex, the *local intersection matrix* of  $\Delta$  at  $q$  is the symmetric matrix  $M_q$  whose rows and columns are indexed by the ridges  $r$  containing  $q$ , and such that the entry in row  $r$  and column  $r'$  is:

$$(M_q)_{r,r'} = \begin{cases} \#\{\text{facets containing } r \text{ and } r'\} & \text{if } r \neq r' \\ -\alpha(v, r) & \text{if } r = r', \end{cases}$$

where  $v$  refers to the vertex of  $r$  not contained in  $q$ . A *tropical complex* is a weak tropical complex  $\Delta$  such that the local intersection matrix  $M_q$  has exactly one positive eigenvalue for each  $(n - 2)$ -dimensional simplex  $q$  of  $\Delta$ .

A regular strictly semistable degeneration  $\mathfrak{X}$  defines a weak tropical complex by augmenting the dual complex of the special fiber with certain intersection numbers. Recall that the dual complex of  $\mathfrak{X}$  has a vertex for each irreducible component of the special fiber and for every connected component in the intersection of  $k + 1$  of these irreducible components, we add a  $k$ -dimensional simplex to the dual complex. If  $s$  is a  $k$ -dimensional simplex of the dual complex, we let  $C_s$  denote the corresponding  $(n - k)$ -dimensional variety in the special fiber of  $\mathfrak{X}$ . Thus, for a vertex  $v$  in a ridge  $r$  of  $\Delta$ ,  $C_v$  and  $C_r$  are a divisor on  $\mathfrak{X}$  and curve, respectively, and so we record the intersection number  $\alpha(v, r) = -\deg C_v \cdot C_r$ .

**Proposition 2.2** (Prop. 2.2 and 2.7 in [Car13]). *A regular strictly semistable degeneration  $\mathfrak{X}$  yields a weak tropical complex  $\Delta$  as above. Moreover,  $\Delta$  is a tropical complex if and only if  $\mathfrak{X}$  is robust in dimension 2.*

We will study linear systems of a divisor on  $X$  by means of the following definition of the specialization to the tropical complex  $\Delta$ . If  $D$  is a divisor on  $X$ , then we let  $\overline{D}$  denote its closure in  $\mathfrak{X}$ . Since  $\mathfrak{X}$  is regular,  $\overline{D}$  is a Cartier divisor on  $\mathfrak{X}$ , and we can take the formal sum over the ridges of  $\Delta$ :

$$\rho(D) = \sum_{r \in \Delta_{n-1}} (\deg \overline{D} \cdot C_r)[r],$$

which we call the *coarse specialization* of  $D$ . In addition, there is a refined specialization, which will be construction in Section 4.

We furthermore define a relationship of linear equivalence on tropical complexes to capture the relationship between specializations of linearly equivalent divisors. First, a *PL function* on a weak tropical complex is a continuous real-valued function which is piecewise linear on each simplex, and has integral slopes, if we identify each simplex with a unimodular simplex. A PL function  $\phi$  defines a formal sum of  $(n - 1)$  polyhedra in  $\Delta$ , denoted  $\text{div}(\phi)$ , by [Car13, Prop. 4.5]. When  $\phi$  is affine linear on each simplex, then its divisor is a linear combination of the ridges of  $\Delta$ , given by the following formula [Car13, Def. 2.4]:

$$(2) \quad \text{div}(\phi) = \sum_{r \in \Delta_{n-1}} \left( \sum_{f \in \Delta_n, r \subset f} \phi(v_{f,r}) - \sum_{v \in r_0} \alpha(v, r) \phi(v) \right) [r],$$

where, in this equation,  $v_{f,r}$  refers to the vertex of  $f$  not contained in the ridge  $r$ . We refer to formal sums arising as the divisor of a PL function as *linearly trivial*, and then we have:

**Proposition 2.3** (Prop. 4.12 in [Car13]). *If  $D$  and  $D'$  are linearly equivalent divisors on the general fiber  $\mathfrak{X}_\eta$  of a strictly semistable degeneration  $\mathfrak{X}$ , then  $\rho(D) - \rho(D')$  is linearly trivial.*

Specializations of divisors are always linear combinations of the ridges of  $\Delta$ , and so the formula (2) for PL functions linear on each simplex is sufficient for Proposition 2.3. However, in our formulation of the specialization inequality, it is important to allow more general PL functions, which will come from linear equivalences on resolutions of  $\mathfrak{X}$  over ramified extensions of  $R$ .

For applications such as Theorem 1.3, it is useful to have a combinatorial criterion for the types of formal sums which can appear from the specialization map  $\rho$ . We can again use the formalism of divisors associated to PL functions to provide such a criterion. Thus, we say that a formal sum of  $(n - 1)$ -dimensional polyhedra is a *Cartier divisor* if it is locally the divisor of a PL function and it is a  $\mathbb{Q}$ -*Cartier divisor* if some positive multiple is Cartier. In this paper, the most important class are formal sums which are  $\mathbb{Q}$ -Cartier away from a set of dimension  $n - 2$ , known as *Weil divisors*, because of the following result:

**Proposition 2.4** (Prop. 4.11 in [Car13]). *Let  $\mathfrak{X}$  be a strictly semistable degeneration of relative dimension at least 2. If  $\mathfrak{X}$  is robust in dimension 2, then for any divisor  $D$  on  $\mathfrak{X}_\eta$ ,  $\rho(D)$  is a Weil divisor.*

### 3. LINEAR SERIES

A *complete linear series* of a Weil divisor  $D$  is the set of all effective Weil divisors linearly equivalent to  $D$ . By *effective*, we mean that all of the coefficients of the divisor are positive. The invariant  $h^0(\Delta, D)$  appearing in Theorem 1.1 is a measure of how large the complete linear series of  $D$  is,

analogous to the dimension of the global sections of  $\mathcal{O}(D)$  on an algebraic variety.

**Definition 3.1.** We say that a point  $p \in \Delta$  is *rational* if its coordinates are rational when if we identify a  $k$ -simplex containing  $p$  with a unimodular simplex in  $\mathbb{R}^k$ . We define  $h^0(\Delta, D)$  to be the cardinality  $m$  of the smallest set of rational points  $p_1, \dots, p_m$  such that there is no effective divisor  $D'$  linearly equivalent to  $D$  such that the  $D'$  contains  $p_1, \dots, p_m$ . If there is no such set of points, then  $h^0(\Delta, D)$  is defined to be infinity.

The rationality condition on the points in Definition 3.1 is needed for technical reasons in the proof of Theorem 1.1 and we expect that it can be dropped without changing the definition.

**Conjecture 3.2.** *The definition of  $h^0$  is equivalent to the analogous definition where the  $p_i$  are allowed to be arbitrary, not necessarily rational, points.*

In the case of a 1-dimensional tropical complex  $\Gamma$ , the quantity  $h^0(\Gamma, D)$  is essentially the same as the rank of the divisor as introduced by Baker and Norine [BN07], and extended to metric graphs in [GK08] and [MZ08], with the exception that our convention differs from theirs by 1. While our definition requires the points to be distinct and rational, the former doesn't affect the definition except when  $\Gamma$  has only one point, and nor does the latter, because in the case of curves, we can prove Conjecture 3.2:

**Proposition 3.3.** *If  $\Gamma$  is a 1-dimensional tropical complex which is not a point and  $r(D)$  is the rank of a divisor  $D$  as in [BN07], then  $h^0(D) = r(D) + 1$ .*

*Proof.* Recall that the rank of  $D$  is the largest number  $r$  such that for any  $r$  points  $p_1, \dots, p_r$  in  $\Gamma$ , the difference  $D - p_1 - \dots - p_r$  is linearly equivalent to an effective divisor  $D'$ . Thus,  $D' + p_1 + \dots + p_r$  is an effective divisor linearly equivalent to  $D$  and it clearly contains the  $p_i$ , so  $h^0(D) \geq r(D) + 1$ . To show the reverse inequality, we assume that there exist points  $p_1, \dots, p_{r(D)+1}$  such that  $D$  is not linearly equivalent to the sum of an effective divisor with  $p_1 + \dots + p_{r(D)+1}$ . In other words, if we write  $|D|$  for the subset of  $\Gamma^d$  which are effective divisors linearly equivalent to  $D$ , then  $|D|$  does not intersect the subset

$$p_1 \times \dots \times p_{r(D)+1} \times \Gamma \times \dots \times \Gamma \subset \Gamma^d.$$

However,  $|D|$  is a closed subset [MZ08, Thm. 6.2]. Therefore, since  $\Gamma$  is not a point, we can perturb the points  $p_i$  slightly and the intersection with  $|D|$  will still be empty. In particular, we can make the  $p_i$  distinct and rational. Thus,  $h^0(D) \leq r(D) + 1$ , so we've proved the proposition.  $\square$

We now give some applications of Theorem 1.1 for 2-dimensional tropical complexes, beginning with the proof of Theorem 1.3. For the example of a 2-dimensional tropical complex which doesn't lift, we take the cylinder depicted in Figure 1, which is a variant of Example 4.3 in [Car15], with the

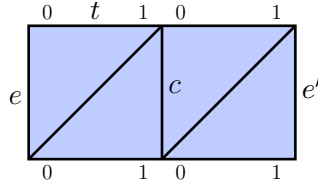


FIGURE 1. The tropical complex  $\Delta$  used in Theorem 1.3 is formed from the given diagram by identifying the edges  $e$  and  $e'$ . For the edges forming the top and bottom circles, the structure constants are indicated by the numbers adjacent to each endpoint. The other edges all have degree 2 and all structure constants on these edges are taken to be 1.

circumference of the cylinder increase to 2 so that the underlying complex is a regular  $\Delta$ -complex.

**Lemma 3.4.** *Let  $\Delta$  be as in Figure 1 and let  $D$  be the sum of the top two edges. Then we claim that any divisor linearly equivalent to  $mD$  is the sum of  $m$  copies of parallel circles, and thus  $h^0(\Delta, mD) = m + 1$ .*

*Proof.* Let  $\phi$  be a PL function on  $\Delta$  such that  $\text{div}(\phi) + mD$  is effective. Let  $C$  be a horizontal circle on the cylinder  $\Delta$  parallel to  $D$ , but not equal to  $D$  or the bottom of the cylinder. Let  $p$  be a point of  $C$  at which the restriction  $\phi|_C$  achieves its maximum. The key point is that the linear functions on a neighborhood  $U$  of  $p$  embed  $U$  in an open subset of  $\mathbb{R}^2$  and  $C$  remains a straight line in this embedding. Since the divisor of  $\phi$  is effective in  $U$ ,  $\phi|_U$  is a function of  $\mathbb{R}^2$ . Thus, the restriction to  $C$ , which is a line segment, is also convex, but  $\phi|_C$  achieves its maximum at  $p$  and so  $\phi|_{C \cap U}$  must be constant. Since  $\phi$  is constant in any neighborhood of a point in  $C$  where  $\phi|_C$  achieves its maximum,  $\phi$  is constant on  $C$ .

Thus,  $\phi$  is a function solely of the vertical coordinate on  $\Delta$ , and  $\phi$  defines a linear equivalence between  $mD$  and  $m$  horizontal circles, possibly not distinct. For any  $m$  points, these circles can be chosen to contain them, but for  $m + 1$  points chosen to have  $m + 1$  distinct heights, no sum of  $m$  circles will contain all of the points. Therefore,  $h^0(\Delta, mD) = m + 1$ , as claimed.  $\square$

*Proof of Theorem 1.3.* As stated above, we take  $\Delta$  to be the tropical complex depicted in Figure 1. The same methods as in [Car15, Ex. 4.3] can be used to compute that the group of ridge divisors up to linear equivalence is isomorphic to  $\mathbb{Z}$ . In particular, the Weil ridge divisors can be characterized by a linear conditions at each vertex of  $\Delta$ , from which the group of Weil ridge divisors can be shown to be isomorphic to  $\mathbb{Z}^4$ . The four generators can be chosen to all be linearly equivalent and so the group of linearly equivalent Weil divisors is  $\mathbb{Z}$ , generated by the equivalence class of

$$D' = [t] + [c] - [e],$$

where edges  $t$ ,  $c$ , and  $e$  are indicated in Figure 1. Moreover,  $2D'$  is linearly equivalent to  $D$ , the sum of the top edges of  $\Delta$ .

Now suppose that  $\mathfrak{X}$  is a degeneration over a discrete valuation ring whose tropical complex is  $\Delta$ . Since  $\Delta$  is a tropical complex and all its maximal simplices are 2-dimensional,  $\mathfrak{X}$  must be robust in dimension 1 and 2 by the text before [Car13, Prop. 2.7] and Proposition 2.2, so Theorem 1.1 applies. The general fiber  $X$  is a smooth proper surface and therefore projective. If we let  $A$  be an ample on  $X$ , then  $h^0(X, \mathcal{O}(mA))$  grows quadratically in  $m$ . On the other hand,  $\rho(A)$  is a ridge divisor by construction, and so  $\rho(2A)$  must be linearly equivalent to  $lD$  for some integer  $l$ . By Lemma 3.4,  $h^0(\Delta, mlD)$  grows linearly in  $m$ . But, Theorem 1.1 gives us the inequality  $h^0(X, \mathcal{O}(mA)) \leq h^0(\Delta, mlD)$ , and so we have a contradiction. Therefore, no such degeneration  $\mathfrak{X}$  exists.  $\square$

The following example shows one case where the inequality in Theorem 1.1 can be sharp for a divisor and all of its multiples.

**Example 3.5.** We consider a degeneration of a quartic surface which will yield a tetrahedron as its dual complex. We start with the variety  $\tilde{\mathfrak{X}} \subset \mathbb{P}_R^3$ , where  $R = \mathbb{C}[[t]]$ , which is defined by the determinant:

$$(3) \quad \begin{vmatrix} xy & (3x^2 + 2y^2 + z^2 - w^2)t \\ 3x^2 + y^2 + 2z^2 + w^2 & zw + t(x^2 + y^2 + z^2 + w^2) \end{vmatrix}$$

The reason for this determinantal form is that we can immediately see that the general fiber  $\tilde{\mathfrak{X}}_K$  contains the scheme defined by the equations in the bottom row of the matrix (3), and one can check that this is a smooth complete intersection and therefore an elliptic curve  $E$ . The curve  $E$  is part of a pencil interpolating between  $E$  and the curve defined by the equations in the top row of (3). Thus, the complete linear series of  $E$  defines a map to  $\mathbb{P}_K^1$  with connected fibers, and so  $h^0(\tilde{\mathfrak{X}}_K, \mathcal{O}(mE)) = m + 1$  for all  $m \geq 0$ .

Now we want to show how Theorem 1.1 gives a sharp bound for this value of  $h^0(\tilde{\mathfrak{X}}_K, \mathcal{O}(mE))$ . We can rewrite (3) as  $xyzw + tf$ , where

$$f = xy(x^2 + y^2 + z^2 + w^2) - (3x^2 + 2y^2 + z^2 - w^2)(3x^2 + y^2 + 2z^2 + w^2),$$

to see that the special fiber of  $\tilde{X}$  is the union of the four coordinate planes. However,  $\tilde{X}$  is not a regular degeneration, but has 24 ordinary double point singularities at the common intersection of the quartic  $f$  and the 6 coordinate lines in  $\mathbb{P}^3$ . We wish to resolve each singularity without introducing any new components in the special fibers, which can be done by blowing up one of the two planes containing the singularity.

As in [Car13, Ex. 2.8], we can obtain a symmetric tropical complex by blowing up one plane at 2 of the 4 singularities along each line and blowing up the other plane at the other 2 singularities, but we pay special attention to the four lines defined by the intersection of one of the planes  $x = 0$  or  $y = 0$  with either  $z = 0$  or  $w = 0$ . Note that when either  $x$  or  $y$  vanishes,  $f$  factors as the product of two quartics, which we write as  $g = 3x^2 + 2y^2 + z^2 - w^2$



and  $h = 3x^2 + y^2 + 2z^2 + w^2$ . At each point of intersection of one of these four lines with  $g = 0$ , we blow up either the  $x = 0$  or  $y = 0$ , as appropriate, and at each point of intersection with  $h = 0$ , we blow up either the  $z = 0$  or  $w = 0$  plane. Along the other 2 coordinate lines, we blow up each containing plane at 2 of the singularities, but chosen arbitrarily.

After these blow ups, the resulting scheme  $\mathfrak{X}$  is a regular semistable degeneration. The special fiber has four components, each of which is the blow up of  $\mathbb{P}^2$  at 6 points, 2 on each coordinate line. The 1-dimensional strata in each component are the union of the strict transforms of the coordinate lines, and one can check that the union of these lines is a big divisor, so  $\mathfrak{X}$  is robust and thus we can apply the specialization inequality from Theorem 1.1. The tropical complex  $\Delta$  is a tetrahedron with all structure constants  $\alpha(v, e)$  equal to 1, using the fact that the strict transform of each coordinate line has self-intersection  $-1$ .

We first calculate the specialization  $\rho(E)$ , where  $E$  is the divisor on  $\mathfrak{X}_K$  defined above. In the original singular degeneration  $\tilde{\mathfrak{X}}$ , the closure  $\bar{E}$  of  $E$  degenerates to the union of two conics, in the  $z = 0$  and  $w = 0$  planes, respectively, each defined by the restriction of the polynomial  $h$ . However, we chose to blow up the  $z = 0$  and  $w = 0$  planes at the intersections of  $h$  with the  $x = 0$  and  $y = 0$  lines, which removes the intersections between  $\bar{E}$  and those lines. Therefore, the only 1-dimensional stratum of  $\mathfrak{X}$  that  $\bar{E}$  intersects is the  $x = y = 0$  line, which it does with multiplicity 2. Thus,  $D = \rho(E)$  is twice one edge of the tetrahedron  $\Delta$ .

Finally, to justify our claim that the specialization inequality is sharp, we need to show that  $h^0(\Delta, mD) = m + 1$  for all non-negative integers  $m$ . For this, we start with a PL function  $\phi$  which is constant on the support of  $D$  and increases with slope 1 on each of the two containing facets, which establishes a linear equivalence between  $D$  and any cycle on the tetrahedron parallel to  $D$ . Moreover, if we ignore both the edge supporting  $D$  and the edge opposite it in  $\Delta$ , we are left with an open cylinder, which has the same theory of linear equivalence as if we removed the top and bottom circles from the cylinder in Figure 1, used in the proof of Theorem 1.3. Therefore, Lemma 3.4 shows any divisor in the complete linear series of  $mD$  consists of  $kD + k'D'$  plus  $l$  cycles parallel to  $D$ , where  $D'$  is twice the edge opposite  $D$  and  $k + k' + l = m$ . As in the proof of Lemma 3.4, such divisors can be chosen to contain any  $m$  points, but not  $m + 1$  points at different distances from  $D$ , and so  $h^0(\Delta, mD) = m + 1$ .  $\square$

The sharpness of the inequality in Example 3.5 depended on our choice of blow ups in constructing the resolution  $\mathfrak{X}$ . In the case of surfaces and higher-dimensional varieties, there is no single minimal regular semistable degeneration, and the choice of the model can affect the bounds from Theorem 1.1.

We close this section by stating a conjectural form of Riemann-Roch for 2-dimensional tropical complexes. Although we have no definition for higher

cohomology, we can assume Serre duality to justify taking  $h^0(\Delta, K_\Delta - D)$  as a replacement for the top cohomology. Here, we take  $K_\Delta$  of any weak tropical complex  $\Delta$  to be the sum  $\sum_r (\deg r - 2)[r]$  over the ridges  $r$  of  $\Delta$ , generalizing the definition for curves. In dimension 1, this approach yielded the Riemann-Roch theorems for graphs and tropical curves [BN07, GK08, MZ08]. In addition, surfaces have a symmetric, bilinear intersection pairing [Car15, Thm. 1.1], which is integral on Cartier divisors. With these ingredients, we conjecture the following, which is a formal analogue to the Riemann-Roch theorem for algebraic surfaces, as it was stated before the introduction of cohomology:

**Conjecture 3.6** (Riemann-Roch). *Let  $D$  be a Cartier divisor on a tropical surface  $\Delta$  and assume that  $K_\Delta$  is also a Cartier divisor. Then*

$$(4) \quad h^0(\Delta, D) + h^0(\Delta, K_\Delta - D) \geq \frac{\deg D \cdot (D - K_\Delta)}{2} + \chi(\Delta),$$

where  $\chi(\Delta)$  is the Euler characteristic of underlying topological space of  $\Delta$ .

**Example 3.7.** We verify Conjecture 3.6 for the tetrahedron  $\Delta$  from Example 3.5 and for any multiple  $mD$ , where  $D$  is twice an edge of the tetrahedron. Since every edge is contained in exactly two facets,  $K_\Delta$  is trivial. Moreover, Example 3.5 exhibited linear equivalences between  $D$  and disjoint divisors, which implies that  $D^2$  has degree zero. Therefore, the right hand side of (4) is  $\chi(\Delta) = 2$  in this example.

Example 3.5 computed that  $h^0(\Delta, mD) = m + 1$  for  $m \geq 0$ . The same method as in that example shows that if  $m > 0$ , then  $h^0(\Delta, K_\Delta - mD) = h^0(\Delta, -mD) = 0$ . (This vanishing of the negative of an effective divisor also holds for a broad class of 2-dimensional tropical complexes, as a consequence of [Car15, Cor. 2.12].) Thus, for any integer  $m$  the left hand side of (4) is:

$$h^0(\Delta, mD) + h^0(\Delta, -mD) = \begin{cases} m + 1 & \text{if } m > 0 \\ 2 & \text{if } m = 0 \\ -m + 1 & \text{if } m < 0, \end{cases}$$

and so the Riemann-Roch inequality is satisfied.

Note that  $\frac{1}{2}D$  still has integral coefficients and so is a  $\mathbb{Q}$ -Cartier divisor. Moreover, one can check that  $h^0(\Delta, \frac{1}{2}D) = 1$  and  $h^0(\Delta, -\frac{1}{2}D) = 0$  and thus the Riemann-Roch inequality is not satisfied for  $D$ . However,  $\frac{1}{2}D$  is not Cartier, which is the reason that the divisor in Conjecture 3.6 is assumed to be Cartier.  $\square$

#### 4. SUBDIVISIONS

In this section, we study subdivisions of a weak tropical complex, which correspond to ramified extensions of the discrete valuation ring  $R$  followed by toroidal resolution of singularities. Here, the key technology is the relationship between toroidal maps and polyhedral subdivisions as in [KKMSD73]. Combinatorially, passing to a degree  $m$  totally ramified extension of the

DVR  $R$  corresponds to scaling the simplices of  $\Delta$  by a factor of  $m$ , and the toroidal resolution is given by a unimodular subdivision of the scaled simplices. The importance of subdivisions for us is as a tool when we define the refined specialization in the next section. First, we check that such subdivisions of a weak tropical complex  $\Delta$  do not change the divisor theory, up to rescaling PL functions.

**Construction 4.1** (Subdivisions). By an *order- $m$  subdivision* of a weak tropical complex  $\Delta$ , we mean the  $\Delta$ -complex which is formed by replacing each simplex of  $\Delta$  by a unimodular integral subdivision of the standard simplex scaled by  $m$ . Any such subdivision has the structure of a weak tropical complex, with the structure constants obtained as follows. For any ridge  $r'$  meeting the interior of a facet of  $\Delta$ , the structure constants can be determined by a method similar to that used for tropical compactifications in [Car13, Constr. 2.3]. In particular, if  $v'_1, \dots, v'_n$  are the vertices of  $r'$ , and  $w_1$  and  $w_2$  contained in the two facets containing  $r'$ , then the midpoint  $(w_1 + w_2)/2$  is contained in the plane spanned by  $r'$ . Thus, we can write:

$$(w_1 + w_2)/2 = c_1 v'_1 + \dots + c_n v'_n$$

for some coefficients  $c_1, \dots, c_n$  with  $c_1 + \dots + c_n = 1$ . We set  $\alpha(v'_i, r') = 2c_i$ .

On the other hand, suppose that  $r'$  is a ridge of  $\Delta'$  which is contained in a degree  $d$  ridge  $r$  of  $\Delta$ . We represent the points of each facet of  $\Delta$  which contains  $r$  by  $(n+1)$  non-negative coordinates with total sum equal to  $m$ . In the  $i$ th such facet, the unique facet of the subdivision  $\Delta'$  containing  $r'$  is the convex hull of  $r'$  and a unique point, whose coordinate is  $(x_{i,1}, \dots, x_{i,n}, 1)$ , by unimodularity. Likewise, the points of  $r$  can be given coordinates consisting of  $n$  non-negative real numbers, also summing to  $m$ . In such coordinates, we represent the  $i$ th vertex  $v_i$  of  $r'$  by the vector  $(y_{i,1}, \dots, y_{i,n})$ . Finally, if  $v_1, \dots, v_n$  are the vertices of  $r$ , we determine the structure constants of  $r'$  by the equation:

$$(5) \quad \begin{pmatrix} \alpha(v'_1, r') \\ \vdots \\ \alpha(v'_n, r') \end{pmatrix} = \begin{pmatrix} y_{1,1} & \cdots & y_{n,1} \\ \vdots & & \vdots \\ y_{1,n} & \cdots & y_{n,n} \end{pmatrix}^{-1} \begin{pmatrix} \alpha(v_1, r) + x_{1,1} + \cdots + x_{d,1} \\ \vdots \\ \alpha(v_n, r) + x_{1,n} + \cdots + x_{d,n} \end{pmatrix}.$$

That these structure constants form a weak tropical complex is verified in the following proposition.  $\square$

**Proposition 4.2.** *The subdivision in Construction 4.1 results in a weak tropical complex.*

*Proof.* We need to show that the  $\alpha(v', r')$  from Construction 4.1 are integral and that they satisfy the required identity (1) for each ridge. We retain the same notation as in Construction 4.1 and we first consider a ridge  $r'$  meeting the interior of a facet of  $\Delta$ . Then the coordinates of  $(w_1 + w_2)/2$  will be half-integers, and so the  $c_i$  are also half-integers, and  $\alpha(v'_i, r') = 2c_i$  is an integer. Moreover, the sum of the  $\alpha(v'_i, r')$  will equal 2, which is the degree of  $r'$  in  $\Delta'$ .

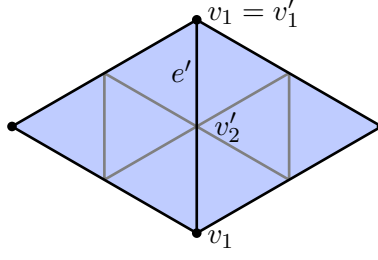


FIGURE 2. The gray lines show a order 2 subdivision of two triangles glued along a common edge. In both the original and subdivided complex,  $\alpha(v, r) = 1$  for all ridges  $r$  which are contained in two facets.

Now consider a ridge  $r'$  of the subdivision  $\Delta'$  contained in a ridge  $r$  of  $\Delta$ . The fact that the vectors  $v'_1, \dots, v'_n$  form the vertices of one simplex of a unimodular triangulation imply that the differences between pairs of the vectors  $(y_{i,1}, \dots, y_{i,n})$  span the integral vectors  $\mathbb{Z}^n$  whose sum is 0. Moreover, each vector satisfies the affine linear equation  $y_{i,1} + \dots + y_{i,n} = m$ , so the span of these vectors includes all vectors in  $\mathbb{Z}^n$  whose sum is a multiple of  $m$ . Thus, to check that the coefficients defined in (5) are integral, it is sufficient to check that the sum of the entries of the vector from that equation is a multiple of  $m$ . Indeed, we know that  $\alpha(v_1, r) + \dots + \alpha(v_n, r) = d$  and that  $x_{i,1} + \dots + x_{i,n} = m - 1$ , so the total sum is  $dm$ . Moreover, this implies that  $\alpha(v'_1, r') + \dots + \alpha(v'_n, r') = d$ .  $\square$

**Example 4.3.** Consider the order 2 subdivision of the complex  $\Delta$  shown in Figure 2. Let  $e'$  be the upper segment of the central edge, with  $v'_1$  the top vertex and  $v'_2$  the middle vertex of the edge. If  $v_1$  is also the top vertex and  $v_2$  is the bottom vertex, then  $(y_{1,1}, y_{1,2}) = (2, 0)$  and  $(y_{2,1}, y_{2,2}) = (1, 1)$ . The vertices adjacent to  $e'$  both have coordinate  $(1, 0, 1)$ . Thus, applying (5), we get:

$$\begin{pmatrix} \alpha(v'_1, e') \\ \alpha(v'_2, e') \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 + 1 + 1 \\ 1 + 0 + 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

More generally, we have  $\alpha(v, e) = 1$  for every internal edge in the subdivided complex  $\Delta'$ .  $\square$

**Remark 4.4.** A subdivision of a tropical complex is also a tropical complex, assuming that all of its maximal cells are facets. In this paper, we will not need this fact because we will deal exclusively with tropical complexes coming from degenerations, in which case the subdivisions will be tropical complexes because the corresponding degenerations are robust in dimension 2, as will be shown in Lemma 6.5. Nonetheless, it is possible to give a combinatorial proof that the local intersection matrices of the subdivision have exactly one positive eigenvalue. The local intersection matrix of the subdivision is equivalent to the matrices derived from local cone complexes in [Car15,

Sec. 2], and these matrices have exactly one positive eigenvalue by [Car15, Lem. 2.7].  $\square$

It is immediate from the definition that a subdivision of a tropical complex is homeomorphic to the original complex. In addition, we now show that the subdivision behaves identically with regards to divisors and linear series.

**Lemma 4.5.** *Let  $\Delta'$  be a order  $m$  subdivision of a weak tropical complex  $\Delta$ , and we identify the realizations of  $\Delta'$  and  $\Delta$  via the natural homeomorphism. Then a function  $\phi$  on  $\Delta$  is linear (respectively PL) if and only if  $m \cdot \phi$  is linear (respectively PL) on  $\Delta'$ . Moreover, the divisor of a PL function  $\phi$  on  $\Delta$  is the same as the divisor of  $m \cdot \phi$  on  $\Delta'$ . In particular, the (Cartier,  $\mathbb{Q}$ -Cartier, Weil) divisors on  $\Delta$  and  $\Delta'$  coincide.*

*Proof.* The agreement of PL functions is clear because that property can be checked on on each simplex. For linear functions, we let  $r$  be a ridge of  $\Delta$  and then claim that the coordinates of the function  $\phi_r$  from Construction 4.2 of [Car13], scaled by  $m$ , and restricted to any neighborhood of a ridge  $r'$  in  $N(r)$  is  $\phi_{r'}$ . For  $r'$  which intersects the interior of a facet of  $\Delta$ , this is because, in this case, the definition of  $\phi_r$  reverses the construction of the structure constants of a weak tropical complex in Construction 4.1.

For  $r'$  contained in  $r$ , we let  $v_i, v'_i, x_{i,j}$  and  $y_{i,j}$  be as in Construction 4.1. Then, we can evaluate:

$$\begin{aligned} \sum_{i=1}^d \phi_r(w'_i) &= \sum_{i=1}^d \frac{1}{m} \left( \phi_r(w_i) + \sum_{j=0}^n x_{i,j} \phi_r(v_j) \right) \\ &= \frac{1}{m} \sum_{j=0}^n \alpha(v_j, r) \phi_r(v_j) + \frac{1}{m} \sum_{i=1}^d \sum_{j=0}^n x_{i,j} \phi_r(v_j) \\ &= \frac{1}{m} \sum_{j=0}^n \left( \alpha(v_j, r) + \sum_{i=1}^d x_{i,j} \right) \phi_r(v_j) \\ &= \sum_{j=0}^n \alpha(v'_j, r') \phi_r(v'_j), \end{aligned}$$

where the last step by the change of variables in (5), together with the linearity of  $\phi_r$  on  $r$ . Therefore,  $\phi_r$  satisfies the defining linear relation of  $\phi_{r'}$ .

We now check that the divisor of a PL function  $\phi$  on  $\Delta$  is the same as the divisor of the scaled function  $m \cdot \phi$  on  $\Delta'$ . The divisor of a PL function can be characterized by the properties listed in Proposition 4.5 of [Car13], so we just need to check that these properties agree for a complex  $\Delta$  and its subdivision  $\Delta'$ . The first two properties are that the divisor is linear in the PL function and local, which are independent of the subdivision. Moreover, the third property is that linear functions are characterized by having trivial divisor, and we've shown that the linear functions on  $\Delta$  and  $\Delta'$  agree, up to scaling. The fourth property normalizes the divisor function via a PL

function supported on a single facet, which also agrees, because when passing to  $\Delta'$  we scale both the coordinates on the complex and the PL function by the same factor of  $m$ .

The fifth and final property normalizes the divisor function on ridges not contained in any facet using functions which are linear in the ordinary sense on the entire ridge. Suppose that  $r$  is a ridge not contained in any facet and has vertices  $v_1, \dots, v_n$ , where we choose coordinates so that  $v_1 = (m, \dots, 0)$  and  $v_n = (0, \dots, 0, m)$ . In these coordinates a function  $\phi$ , which is affine linear in the ordinary sense on  $r$ , can be written as  $\phi(x) = c^T x$ , where  $c$  and  $x$  are  $n$ -dimensional column vectors. Thus, the divisor of  $\phi$  along  $r$  according to [Car13, Prop. 4.5(v)] is:

$$-\sum_{i=1}^n \phi(v_i) \alpha(v_i, r) = -\sum_{i=1}^n m c_i \alpha(v_i, r) = -m c^T (\alpha(v_1, r), \dots, \alpha(v_n, r))^T$$

On the other hand, if  $r'$  is a unimodular simplex subdividing  $r$  with vertices  $v'_i$  having coordinate  $y_i = (y_{i,1}, \dots, y_{i,n})$  as in Construction 4.1, then the coefficient of  $m\phi$  along  $r'$  is:

$$-\sum_{i=1}^n m \phi(v'_i) \alpha(v'_i, r') = -\sum_{i=1}^n m (c^T Y)_i \alpha(v'_i, r') = -m c^T Y \begin{pmatrix} \alpha(v'_1, r') \\ \vdots \\ \alpha(v'_n, r') \end{pmatrix},$$

where  $Y$  is the matrix with columns  $y_i$ . These two coefficients agree by the defining equation (5) in the construction of the subdivision.

An immediate consequence is that a formal sum of  $(n-1)$ -dimensional polyhedra in  $\Delta$  is a Weil, Cartier, or  $\mathbb{Q}$ -Cartier divisor if and only if it is in  $\Delta'$ .  $\square$

On the other hand, the subdivisions of Construction 4.1 mirror what happens for toroidal resolutions of ramified base changes.

**Lemma 4.6.** *Let  $\mathfrak{X}$  be a regular semistable degeneration with dual complex  $\Delta$ . If  $\Delta'$  is an order  $m$  subdivision of  $\Delta$  and  $R'$  is a totally ramified degree  $m$  extension of  $R$ , then the corresponding resolution  $\mathfrak{X}'$  of  $\mathfrak{X} \times_{\mathrm{Spec} R} \mathrm{Spec} R'$  has  $\Delta'$  as its weak tropical complex.*

*Proof.* We first describe the pullback of a divisor  $C_v$  from  $\mathfrak{X}$  to  $\mathfrak{X}'$ . If  $v'$  is a vertex of  $\Delta'$ , contained in a  $k$ -dimensional simplex  $s$  of  $\Delta$ , then we can express the coordinate of  $v'$  by a vector  $(x_0, \dots, x_k)$  with  $x_0 + \dots + x_k = m$ . If the vertices of  $s$  are  $v_0, \dots, v_k$ , then we claim that the coefficient of  $C_{v'}$  in the pullback of  $C_{v_i}$  is  $x_i$ . The reason is that the defining equation of  $C_v$  corresponds to the vector  $(0, \dots, 1, \dots, 0)$  in lattice dual to the lattice of valuations in which  $v'$  lives.

We now adopt the notation of Construction 4.1. We compute the intersection number  $\pi^{-1}(C_{v_i}) \cdot C_{F'}$  in two different ways. First, using the discussion in the previous paragraph, we can represent  $\pi^{-1}(C_{v_i})$  as a linear combination of divisors in  $\mathfrak{X}'$ . The ones which intersect  $C_{F'}$  correspond to the vertices

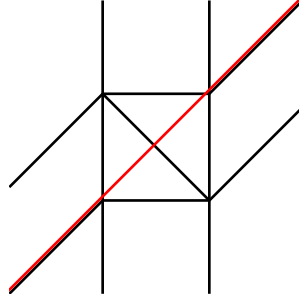


FIGURE 3. The black lines give a subdivision of the plane giving a semistable family whose general fiber is  $\mathbb{P}^1 \times \mathbb{P}^1$ . The red line is the tropicalization of a line in one of the rulings whose specialization to the tropical complex consists the sum of the outside edges of the square minus the diagonal edge. In particular, the specialization is not effective.

of  $F'$ , together with the vertices in neighboring simplices. All together, these give:

$$x_{1,i} + \cdots + x_{d,i} - \alpha(v'_0, F')y_{0,i} - \cdots - \alpha(v'_n, F')y_{n,i}$$

On the other hand, using the projection formula,  $\pi^{-1}(C_{v_i}) \cdot C_{F'} = C_{v_i} \cdot \pi(C_{F'})$ , and we know that  $\pi_*(C_{F'})$  is  $C_F$ , so the intersection number is  $-\alpha(v_i, F)$ . Putting these equations together for all  $i$ , we get:

$$\begin{pmatrix} y_{0,0} & \cdots & y_{n,0} \\ \vdots & & \vdots \\ y_{0,n} & \cdots & y_{n,n} \end{pmatrix} \begin{pmatrix} -\alpha(v'_0, F') \\ \vdots \\ -\alpha(v'_n, F') \end{pmatrix} + \begin{pmatrix} x_{1,0} + \cdots + x_{d,0} \\ \vdots \\ x_{1,n} + \cdots + x_{d,n} \end{pmatrix} = \begin{pmatrix} -\alpha(v_0, F) \\ \vdots \\ -\alpha(v_n, F) \end{pmatrix}.$$

We can solve for the  $\alpha(v'_i, F')$  and we get (5).  $\square$

## 5. REFINED SPECIALIZATION

In this section, we define a refined specialization map from divisors on the general fiber of  $\mathfrak{X}$  to Weil divisors on the tropical complex  $\Delta$ . Given a divisor  $D$ , we first choose a ramified extension of the discrete valuation ring  $R$ , followed by a toroidal resolution of singularities  $\mathfrak{X}'$  such that if  $D'$  is the pullback of  $D$  to the general fiber of  $\mathfrak{X}'$ , the closure  $\overline{D}'$  does not contain any strata of the special fiber of  $\mathfrak{X}'$ . Then, the refined specialization of  $D$  is  $\rho(D')$ , which can be considered as a divisor either on the subdivision  $\Delta'$  or on  $\Delta$  by the results in the previous section. The refined specialization is the higher-dimensional analogue of the map  $\tau_*$  from [Bak08, Sec. 2C]. As stated, in the introduction, the purpose of the refined specialization is that the refined specialization of an effective divisor is always effective, which is not necessarily true for the specialization  $\rho$ .

**Example 5.1.** We consider the plane itself as the tropicalization of  $\mathbb{G}_m^2 \subset \mathbb{G}_m^2$ . However, we choose the subdivision of the plane shown in Figure 3,

which gives a semistable family whose general fiber is  $\mathbb{P}^1 \times \mathbb{P}^1$  and such that the dual complex of the special fiber is the triangulated square  $\Delta$  which is the bounded part of the subdivision. Moreover, using the criterion of [Car13, Prop. 3.8], we check that the locally closed strata of this family are affine.

Now consider a line in one of the rulings of  $\mathbb{P}^1 \times \mathbb{P}^1$  whose tropicalization is the red line in Figure 3. The specialization of this line to  $\Delta$  is the sum of four outside edges of the square minus the inner diagonal. In particular, this specialization is not effective and it does not agree with the tropicalization. However, the specialization is linearly equivalent to an effective divisor, namely the intersection of the red tropicalization with  $\Delta$ .  $\square$

In Example 5.1, it was easy to guess the refined tropicalization. In general, we need to show that there exists a subdivision of the weak tropical complex  $\Delta$  which will give the refined specialization according to the definition at the beginning of this section, which is the content of the following lemma. The construction is essentially a toroidal version of the partial resolution of a hypersurface singularity coming from its Newton polyhedron.

**Lemma 5.2.** *Let  $D$  be an effective divisor on the general fiber  $\mathfrak{X}_\eta$  of a degeneration  $\mathfrak{X}$ . Then, there exists a finite ramified extension  $R'$  of  $R$  and a toroidal resolution of singularities  $\mathfrak{X}'$  of  $\mathfrak{X} \otimes_R R'$  such that  $\overline{D}'$  doesn't contain any of the strata of  $\mathfrak{X}'$ , where  $\overline{D}'$  is the closure in  $\mathfrak{X}'$  of the pullback of  $D$  to the general fiber  $\mathfrak{X}'_\eta$ .*

*Proof.* We let  $s$  be any  $k$ -dimensional simplex of  $\Delta$ . Let  $A$  be the local ring of  $\mathfrak{X}$  at the generic point of  $C_s$ . By assumption,  $A$  is a regular ring with maximal ideal generated by  $x_0, \dots, x_k$ , each of which is the local equation of one of the components of  $\mathfrak{X}_0$  containing  $C_s$ . The completion  $\hat{A}$  of  $A$  is isomorphic to  $S[[x_0, \dots, x_k]]/\langle x_0 \cdots x_k - \pi \rangle$ , where  $S$  is a complete, local extension of  $R$  and  $\pi$  is a uniformizer in  $S$ . We let  $f \in \hat{A}$  be the local equation of  $\overline{D}$  in the completed local ring  $\hat{A}$ . We construct the local Newton polyhedron from  $f$  as follows. For each term  $s_a x_0^{a_0} \cdots x_k^{a_k}$  in the summation representation of  $f$ , we translate the positive orthant in  $\mathbb{R}^{k+1}$  by  $(a_0 + \text{val}(s_a), \dots, a_k + \text{val}(s_a))$ . Then, the local Newton polyhedron of  $f$  is the convex hull of the translated orthants coming from all terms of  $f$ . If  $f$  is not a unit, this convex hull will not contain the origin. In this case, we take the dual polyhedron of the local Newton polyhedron and project its bounded faces to the unit  $k$ -dimensional simplex along lines through the origin. This yields a rational polyhedral subdivision of  $s$ .

We now claim that these subdivisions are compatible so that taken together they form a rational polyhedral subdivision of  $\Delta$ . It is sufficient to show compatibility between the subdivision of  $s$  and of one of its faces  $s'$ , which we can also take to be the face indexed by  $k$ . Thus, in the local ring  $A$ , the component  $C_{s'}$  is defined by the ideal  $\langle x_0, \dots, x_{k-1} \rangle$ , and so its local ring is the localization of  $A$  at this ideal. We consider the image of  $f$ , the local equation of  $D$  in the tensor product  $\hat{A} \otimes_A A_{\langle x_0, \dots, x_{k-1} \rangle}$ . We can now regroup



the terms of  $f$  as:

$$\sum_{a_0, \dots, a_{k-1}} \left( \sum_{a_k} s_a x_k^{a_k} \right) x_0^{a_0} \cdots x_{k-1}^{a_{k-1}}.$$

This expression also holds in the completion of the local ring of  $C_{\partial_k s}$ , so the local Newton polyhedron of  $f$  along  $C_{\partial_k s}$  is the projection of the polyhedron along  $C_s$ . Thus the dual of the Newton polyhedron is the restriction of the dual, which shows the desired compatibility.

Now we have a polyhedral subdivision of  $\Delta$ . If  $m$  is the least common multiple of the denominators of the vertices of the subdivision, then it corresponds to a ramified degree  $m$  extension  $R'' \supset R$  followed by a toroidal map  $\mathfrak{X}'' \rightarrow \mathfrak{X} \times_R R''$ . Although  $\mathfrak{X}''$  is not semistable, the closure  $\overline{D}''$  of the pullback of  $D$  to  $\mathfrak{X}''$  does not contain any of the strata. By [KKMSD73, Thm. III.4.1], there exists a further extension and toroidal map  $\mathfrak{X}' \rightarrow \mathfrak{X}'' \times_{R''} R'$  such that  $\mathfrak{X}'$  is semistable. Since each stratum of  $\mathfrak{X}'$  maps to a stratum of  $\mathfrak{X}''$ , the pullback of  $\overline{D}''$  to  $\mathfrak{X}'$  doesn't contain any strata of the special fiber either. This implies that the pullback of  $\overline{D}''$  is equal to the closure of the pullback of  $D$ .  $\square$

After the subdivision given by Lemma 5.2, the specialization of  $D$  will be effective, because all of the intersections between  $\overline{D}$  and curves  $C_r$  will be proper, and thus the coefficients making up the definition of the specialization will all be non-negative. In addition to the above construction, the refined specialization has a more intrinsic interpretation, at least set-theoretically, as the projection of the Berkovich analytification of the divisor to the skeleton defined by the degeneration. Here, we are using the deformation retract from the Berkovich analytification  $\mathfrak{X}_\eta^{\text{an}}$  to the dual complex of  $\mathfrak{X}$ . Such a retract is defined in a more general setting in [Ber99] and in the specific case of regular semistable degenerations in [Nic11, Sec. 3.3].

*Proof of Proposition 1.2.* By Lemma 5.2, we can assume that our degeneration  $\mathfrak{X}$  is chosen such that  $\overline{D}$  doesn't contain any of the closed strata of  $\mathfrak{X}$ . Because projection of  $\mathfrak{X}_\eta^{\text{an}}$  to the skeleton is compatible with the specialization map, we know that the image of  $D^{\text{an}}$  is contained in those simplices  $s$  such that  $\overline{D}$  intersects  $C_s$ . By hypothesis,  $\overline{D}$  doesn't intersect any 0-dimensional strata  $C_f$  and whenever it intersects a curve  $C_r$  for some ridge  $r$ , it does so properly and thus  $r$  has positive coefficient in the specialization of  $D$ . Therefore, the projection of the analytification is contained in the union of the support of the refined specialization of  $D$ , together with some simplices of codimension at least 2.

It now only remains to show that the projection map is surjective onto the ridges  $r$  in the support of the specialization. For this we suppose that  $r$  is a ridge such that  $\overline{D}$  intersects  $C_r$  and let  $x$  be a point in the intersection. In a neighborhood of  $x$ , we can take local defining equations for the divisors containing  $C_r$  and use them to define a map to  $\text{Spec } S$  where

$S = R[x_1, \dots, x_n]/\langle x_1 \cdots x_n - \pi \rangle$ . As in [Nic11, Sec. 3.3, Case 1], the  $(n-1)$ -dimensional simplex, which we identify with  $r$  embeds in the analytification of  $\text{Spec } S$  as the skeleton. Since the fiber over  $x_1 = \cdots = x_n = 0$  is finite, the map from  $\overline{D}$  to  $\text{Spec } S$  is dominant and so by Proposition 3.4.6(7) of [Ber90], we can lift any norm corresponding to a point in this skeleton to a point in the analytification of  $D$ . Since the projection onto the skeleton of  $\mathfrak{X}_\eta$  is defined in terms of the norms of the  $x_i$ , we've produced a norm in the analytification of  $\overline{D}$  which projects onto any point of  $r$ .  $\square$

The image of the projection in Proposition 1.2 can be larger than the refined specialization whenever  $\overline{D}$  meets a component  $C_v$ , but doesn't meet any curves  $C_r$  contained in  $C_v$ .

**Example 5.3.** Suppose  $\mathfrak{X}$  is any degeneration of dimension at least 2 and  $x$  is a point of  $\mathfrak{X}_\eta$  whose closure in  $\mathfrak{X}$  is contained in  $C_v \setminus D_v$  for some vertex  $v$ . Then, the blow-up  $\mathfrak{X}'$  of  $\mathfrak{X}$  at the closure of  $x$  is a regular semistable degeneration and the specialization of the exceptional divisor  $E$  of the blow-up is trivial because its closure doesn't intersect any curve  $C_r$  in  $\mathfrak{X}'_\eta$ . However, the analytification of  $E$  is not empty and by the compatibility with specialization noted in the proof of Proposition 1.2, its projection to  $\Delta$  must be the single point  $v$ , which has codimension  $n \geq 2$ .  $\square$

## 6. PROOF OF THE SPECIALIZATION INEQUALITY

In this section, we finish the proof of Theorem 1.1. The main remaining ingredient is to show that the refined specialization always preserves point containment, at least for appropriately chosen points. Example 5.3 shows that even for robust degenerations, the specialization of a non-trivial divisor can be trivial, which shows the importance of allowing for a choice of point. However, the hypothesis that the locally closed strata are affine rules out the possibility of non-trivial effective divisors with trivial specialization, as in Example 5.3.

**Proposition 6.1.** *Suppose that  $\mathfrak{X}$  is a degeneration whose locally closed strata of dimension  $m$  are affine for all  $2 \leq k \leq m$ . If  $E$  is a divisor in  $\mathfrak{X}_\eta$  and its closure  $\overline{E}$  intersects some stratum  $C_s$  of the special fiber of  $\mathfrak{X}$  and  $C_s$  has dimension  $m$ , then  $\overline{E}$  intersects the curve  $C_r$  for some ridge  $r$  containing  $s$ .*

*Proof.* The proof is by induction the dimension  $m$ . If  $m = 1$ , then  $s$  is a ridge and we're done. Otherwise, by assumption,  $C_s \setminus D_s$  is affine, and affine varieties contain no complete subvarieties of positive dimension. However,  $\overline{E} \cap C_s$  is a divisor in  $C_s$  and so of dimension  $m - 1 \geq 1$ . Thus,  $\overline{E}$  must intersect  $D_s$  and thus  $C_{s'}$  for some  $(n - m + 1)$ -simplex  $s'$  containing  $s$ . Applying the inductive hypothesis, we get that  $\overline{E}$  intersects  $C_r$  for some ridge  $r$ .  $\square$

Proposition 6.1 gives us a tool for proving that the refined specialization contains a given vertex of the dual complex, and together with Lemma 6.5 below, it can be used to give a proof of Theorem 1.1 for degenerations whose locally closed strata are affine. However, Theorem 1.1 applies not only to degenerations where the locally closed strata are affine, but also to those which are only robust in the top two dimensions. In such cases, the conclusion of Proposition 6.1 doesn't necessarily hold, as shown by Example 5.3. However, we can still ensure that the refined specialization preserves point containment, so long as the point is chosen sufficiently generically.

**Lemma 6.2.** *Assume that  $R$  is complete and that  $\mathfrak{X}$  is robust in dimension  $n - 1$  and  $n$  and that the locally closed strata of dimension at most  $n - 2$  are affine. Then for any vertex  $v$  in  $\Delta$ , there exists a  $K$ -point  $x$  in  $\mathfrak{X}$  such that for any effective divisor  $E \subset \mathfrak{X}_\eta$  containing  $x$ , the refined specialization of  $E$  contains  $v$ .*

Since the crux of this lemma is using the robustness hypothesis, we first prove a couple of results about big divisors. We begin with an application of Bertini's theorem to the connectedness of big divisors.

**Lemma 6.3.** *Any big Cartier divisor has a unique connected component which is itself a big Cartier divisor.*

*Proof.* Suppose that  $E$  is a big divisor on a variety  $X$ . Then we let  $\tilde{X}$  be the blow-up of the base locus and let  $\mathcal{O}_{\tilde{X}}(1)$  be the relative ample divisor. Let  $\tilde{E}$  be the pullback of  $E$  and then  $\tilde{E}(-1)$  defines a regular morphism from  $\tilde{X}$  to  $\mathbb{P}^N$ . By Bertini's theorem [Jou83, Thm. 7.1], the support of  $\tilde{E}(-1)$  is connected. Therefore, the base locus of  $E$  contains all but one connected component of  $E$ . By taking just that connected component, we've only removed fixed divisors, and so we still have a big divisor, which is the desired statement.  $\square$

**Lemma 6.4.** *Suppose that  $E$  is a big Cartier divisor on a variety  $X$ . If  $\phi$  is any morphism from  $X$  to  $\mathbb{P}^N$ , then either  $\phi(E) = \phi(X)$  or  $\phi$  is generically finite on some irreducible component of  $E$ .*

*Proof.* Suppose, for contradiction, that  $E$  is a big divisor on  $X$  and  $\phi: X \rightarrow \mathbb{P}^N$  is a morphism which is not generically finite on any component of  $E$  and such that  $\phi(X) \neq \phi(E)$ . We first consider the case that  $\phi$  is generically finite on  $X$ . Then,  $\dim(X) = \dim \phi(X) \geq \dim \phi(E) + 2$ , so for any point in  $\phi(X) \setminus \phi(E)$ , we can take an intersection with a general linear space to obtain a curve through  $\phi(X)$  which doesn't intersect  $\phi(E)$ . By taking the preimage, we get a curve passing through any sufficiently general point in  $X$ , which does not intersect  $E$ . Since  $E$  is big, such curves can't exist, and so we get a contradiction.

Second, we suppose that  $\phi$  is not generically finite on  $X$ , but still that  $\phi(X) \neq \phi(E)$ . Then for any point in  $\phi(X) \setminus \phi(E)$ , the fiber of  $\phi$  is at least 1-dimensional and we can choose a curve in this fiber. Again, we've

constructed curves in  $X$  which pass through sufficiently general points, and which do not intersect  $E$ , so we've again contradicted our assumption that  $E$  is big.  $\square$

*Proof of Lemma 6.2.* Because  $R/\mathfrak{m}$  is algebraically closed, we can choose a general point  $x_0$  in  $C_v \setminus D_v$ . By general, we mean that it lies outside of finitely many closed subvarieties, which don't depend on  $E$  and will be made explicit in the course of the proof. By Hensel's Lemma, we can lift the point  $x_0$  to an  $R$ -point of  $\mathfrak{X}$ , which will give the desired point  $x$ .

Now let  $E$  be any divisor of  $\mathfrak{X}_K$  containing  $x$  and we first want to show that its closure  $\overline{E}$  in  $\mathfrak{X}$  intersects some 1-dimensional stratum  $C_r$ . If the dimension  $n$  is 1, then this is immediate, so we assume that  $n \geq 2$ . By assumption, some multiple of  $D_v$  defines a rational map  $\phi_v: C_v \dashrightarrow \mathbb{P}^N$  which is birational onto its image. We can assume that our choice of  $x_0$  was in the locus where  $\phi_v$  defines an isomorphism onto its image. Since  $\overline{E}$  contains  $x_0$ , the restriction of  $\phi_v$  to  $E_0 = \overline{E} \cap C_v$  must also be birational onto its image. Since this restriction is defined by the pullback of the same multiple of  $D_v$ , we see that  $D_v \cap E_0$  is a big divisor on  $E_0$ .

By Lemma 6.3, we can take  $B$  to be a connected component of  $D_v \cap E_0$  which is a big divisor on  $E_0$ . Since  $B$  is connected, it is either contained in a single stratum  $C_e$  for some edge  $e$  or it meets some  $C_t$ , where  $t$  is a 2-simplex of  $\Delta$ . If  $n = 2$ , then  $C_e$  is a 1-dimensional stratum intersecting  $\overline{E}$ , which is what we wanted to show. So, we assume that  $n \geq 3$  and, for the moment, we also assume that  $B$  is contained in a single  $C_e$ .

By assumption,  $D_e$  is a big divisor in  $C_e$ , so some multiple of  $D_e$  defines a map  $\phi_e: C_e \dashrightarrow \mathbb{P}^M$ , birational onto its image. Since this morphism is just defined by a collection of rational functions on  $C_e$ , we can extend it to a rational map from  $C_v$  to  $\mathbb{P}^M$ , also denoted by  $\phi_e$ . The image  $\phi_e(C_v)$  contains the image of  $C_e$ , so it is at least  $(n-1)$ -dimensional. Therefore, a general fiber of  $\phi_e$  is at most 1-dimensional, so we can assume that  $\phi_e^{-1}(\phi_e(x_0))$ , the fiber containing  $x_0$ , has dimension at most 1. Thus,  $\phi_e(E_0)$  must be at least  $(n-2)$ -dimensional. We apply Lemma 6.4 to the restriction  $\phi_e|_{E_0}$ , and in either of that lemma's two cases,  $\phi_e(B)$  is also at least  $(n-2)$ -dimensional. Since  $\phi_e(B)$  is positive dimensional, it intersects any hyperplane in  $\mathbb{P}^M$ , which shows that  $B$  must intersect  $D_e$ , and thus  $C_t$  for some 2-simplex  $t$ .

At this point, we've shown that if  $n \leq 3$ , then  $\overline{E}$  intersects  $C_r$  for some ridge  $r$  containing  $v$ . If  $n > 3$ , then  $\overline{E}$  at least intersects  $C_t$  for some 2-simplex  $t$  containing  $v$ , but we can then apply Proposition 6.1 to show that  $\overline{E}$  intersects some  $C_r$  for some ridge containing  $v$  in this case as well.

We now consider the resolution of the base change  $\mathfrak{X}'$  produced by Lemma 5.2. There are two cases, depending on whether or not there exists a facet  $f$  containing  $r$  such that  $\overline{E}$  contains the point  $C_f$ . If there does, then we consider the component  $C'_v$  in  $\mathfrak{X}'$  corresponding to  $v$ , which maps birationally onto  $C_v$ . By the properness of the resolution, there must be a point in  $C'_v \cap \overline{E}'$  mapping to the point  $C_f$ , where  $\overline{E}'$  is the closure of  $E$  in  $\mathfrak{X}'$ . However, the

fiber of  $C_f$  in the resolution  $\mathfrak{X}'$  is a union of toric varieties intersecting along their toric boundaries and so by Proposition 6.1,  $\overline{E}'$  must intersect one of the 1-dimensional strata in this fiber. By Lemma 5.2, this intersection is proper, so it gives a positive coefficient to some ridge containing  $v$ .

On the other hand, suppose that  $\overline{E}$  does not intersect  $C_f$  for any facet  $f$  containing  $r$ . We've assumed that  $D_r$  is a big divisor on  $C_r$ , so it is a non-trivial divisor. In other words, there is at least one stratum  $C_f$  properly contained in  $C_r$ . Since  $\overline{E}$  does not contain any  $C_f$ , it must intersect  $C_r$  properly, and thus with positive intersection number. Now we choose a ridge  $r'$  containing  $v$  in the subdivision of  $r$  and then the corresponding stratum  $C'_{r'}$  of  $\mathfrak{X}'$  maps surjectively onto  $C_r$ . By the projection formula, this implies that the intersection number of  $\pi^{-1}(\overline{E})$  with  $C'_{r'}$  is positive, where  $\pi$  is the map from  $\mathfrak{X}'$  to  $\mathfrak{X}$ . Of course,  $\pi^{-1}(\overline{E})$  is not the same as  $\overline{E}'$ , but also includes any components of the special fiber whose image is contained in  $\overline{E}$ . However, the only such components which affect the intersection number will be those which meet  $C'_{r'}$ , in which case the projection of the component onto  $\mathfrak{X}$  must be either  $C_r$  or a point  $C_f$  contained in  $C_r$ . However, we've assumed that in neither of these can be contained in  $\overline{E}$ , so  $\overline{E}' \cdot C'_{r'} = \pi^{-1}(\overline{E}) \cdot C'_{r'}$  is positive, which shows that the ridge  $r'$  containing  $v$  occurs with positive coefficient in the refined specialization, as desired.  $\square$

We now have one lemma remaining before finishing the proof. While Lemma 6.2 shows how to find a divisor whose refined specialization contains a given vertex, our definition of  $h^0$  allowed arbitrary rational points. We can turn arbitrary rational points into vertices by choosing an appropriate subdivision, but we need to check that such a subdivision preserves the robustness and affineness properties of the degeneration.

**Lemma 6.5.** *Let  $\mathfrak{X}$  be a tropical complex and let  $\mathfrak{X}'$  be any toroidal resolution of a base change of  $\mathfrak{X}$  to a totally ramified extension of  $R$ . Let  $m$  be an integer. If the  $k$ -dimensional locally closed strata of  $\mathfrak{X}$  are affine for all  $k \leq m$ , then the same is true for the  $k$ -dimensional locally closed strata in  $\mathfrak{X}'$  when  $k \leq m$ . If the  $\mathfrak{X}$  is robust in dimension  $k$  for all  $k \leq m$ , then  $\mathfrak{X}'$  is also robust in dimensions  $k \leq m$ .*

*Proof.* Let  $\mathfrak{X}'$  be a toroidal resolution of a ramified extension of  $\mathfrak{X}$ . We first consider the case that the  $k$ -dimensional locally closed strata of  $\mathfrak{X}$  are affine for  $k \leq m$ . Suppose that  $C_{s'}$  is a stratum of  $\mathfrak{X}'$  which has dimension  $k' \leq m$ . Then the image of  $C_{s'}$  in  $\mathfrak{X}$  is the stratum  $C_s$ , where  $s$  is the minimal simplex of  $\Delta$  which contains  $s'$ . Therefore,  $k$ , the dimension of  $C_s$  satisfies  $k \leq k' \leq m$ , and so  $C_s \setminus D_s$  is affine by assumption. From the construction of the toroidal resolution, we know that  $C_{s'}$  will be a toric variety bundle over  $C_s$  and  $C_{s'} \setminus D_{s'}$  will be a  $\mathbb{G}_m^{k'-k}$ -bundle over  $C_s \setminus D_s$ . Therefore, the map from  $C_{s'} \setminus D_{s'}$  to  $C_s \setminus D_s$  is affine and so  $C_{s'} \setminus D_{s'}$  is affine as well. We conclude that the  $k$ -dimensional locally closed strata of  $\mathfrak{X}'$  are affine for  $k \leq m$ .

We now consider the case that  $\mathfrak{X}$  is robust in dimensions  $k$  for  $k \leq m$ . Let  $C_{s'}$  and  $C_s$  be strata of  $\mathfrak{X}'$  and  $\mathfrak{X}$  of dimensions  $k'$  and  $k$  as in the previous paragraph. By assumption, some multiple of the divisor  $D_s$  defines a birational map  $\phi$  from an open subset of  $C_s$  to  $\mathbb{P}^N$ . We let  $\overline{\phi(C_s)}$  denote the closure of the image of this map and let  $Y$  denote the image of  $C_s \setminus D_s$ . Since  $Y$  is the complement of a hyperplane section in a projective variety,  $Y$  is affine.

Now, we consider  $C_{s'}$ , which, as above, is a toric variety bundle over  $C_s$ , and the toric variety is described by the star of  $s'$  in  $s$ . We can choose some positive multiple of the boundary of this toric variety which contains a spanning set of the characters of the torus, i.e. defines a birational map to projective space. Let  $\mathcal{L}$  be the corresponding line bundle on  $C_{s'}$ . Explicitly,  $\mathcal{L}$  is the line bundle associated to a multiple of the sum of the divisors corresponding to the simplices of  $\Delta'$  containing  $s'$ , which are also contained in  $s$ . We consider the coherent sheaf on  $Y$ ,  $(\phi \circ \pi)_*(\mathcal{L})$ , where  $\pi: C_{s'} \rightarrow C_s$  is the restriction of the map from  $\mathfrak{X}'$  to  $\mathfrak{X}$ .

We can find an open set  $U$  of  $Y$  such that  $\phi$  is an isomorphism on  $U$  and  $(\phi \circ \pi)^{-1}(U)$  is the trivial toric variety bundle over  $\phi^{-1}(U) \cong U$ . Furthermore, we can assume that  $U$  is the complement of the variety defined by some element  $f$  of the global sections of  $Y$ , since  $Y$  is affine. On  $U$ , the push-forward  $(\phi \circ \pi)_*(\mathcal{L})$  is a free sheaf whose generators can be identified with the sections on the toric variety. In particular, these sections define a rational map from  $(\phi \circ \pi)^{-1}(U)$  to projective space which is birational on each fiber of  $\pi$ . Since  $Y$  is affine, we lift these sections from  $U$  to  $Y$  after multiplying by a sufficiently large power of  $f$ . If we regard  $f$  as a section of  $\mathcal{O}(\ell D_s)$  for sufficiently large  $\ell$ , then what we've found are sections of  $\mathcal{L} \otimes \pi^{-1}(\mathcal{O}(\ell D_s))$  which define an embedding of the torus for a generic fiber of  $\pi$ . Combining these with the pullbacks of the sections defining  $\phi$ , we get a birational map from  $C_{s'}$ , and therefore  $D_{s'}$  is a big divisor and  $\mathfrak{X}'$  is robust in dimensions  $k \leq m$ .  $\square$

*Proof of Theorem 1.1.* By base changing, we can assume that  $R$  is complete without changing  $h^0(\mathfrak{X}_\eta, D)$  or  $\Delta$ . Let  $r$  be an integer less than  $h^0(\mathfrak{X}_\eta, \mathcal{O}(D))$  and let  $p_1, \dots, p_r$  be any  $r$  rational points in  $\Delta$ . We want to show that there exists an effective divisor linearly equivalent to  $\rho(D)$  containing these points. Let  $m$  be the least common denominator of the coordinates of the points  $p_i$ . We rescale the simplices by  $m$  and subdivide them using the “regular subdivision” of [KKMSD73, Thm. III.2.22], and now the points  $p_i$  are vertices. We replace  $\mathfrak{X}$  with the corresponding toroidal resolution of a ramified extension of  $R$ . It will suffice to work with this subdivision by Lemmas 4.5, 4.6, and 6.5.

Using Lemma 6.2, we can choose points  $x_1, \dots, x_r$  in  $\mathfrak{X}_\eta$  corresponding to  $p_1, \dots, p_r$  respectively. Since each  $x_i$  is  $K$ -rational, then vanishing on  $x_i$  imposes one linear condition on the sections  $H^0(\mathfrak{X}_\eta, \mathcal{O}(D))$ . We've assumed that  $r$  is less than the dimension of this vector space, so there exists a

non-zero section of  $H^0(\mathfrak{X}_\eta, \mathcal{O}(D))$  defining a divisor  $D'$  which contains all of the  $x_i$ . Let  $D''$  be the refined specialization of  $D'$ . Then  $D''$  is effective by Lemma 5.2 and contains the points  $p_1, \dots, p_r$  by Lemma 6.2. By [Car13, Prop. 3.10],  $D''$  is linearly equivalent to  $\rho(D)$ . Therefore,  $h^0(\Delta, \rho(D))$  is greater than  $r$ , which establishes the desired inequality.  $\square$

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